BUBBLE STABILIZED DISCONTINUOUS GALERKIN METHOD FOR STOKES' PROBLEM

ERIK BURMAN AND BENJAMIN STAMM

Abstract. We propose a low order discontinuous Galerkin method for incompressible flows. Stability of the discretization of the Laplace operator is obtained by enriching the space element wise with a non-conforming quadratic bubble. Several possible pressure spaces that lead to uniformly stable velocity pressure pairs are proposed. We prove optimal convergence estimates and local conservation of both mass and linear momentum independent of numerical parameters.

1. Introduction

Discontinuous Galerkin (DG) methods for incompressible flow has been studied in Hansbo and Larson [11] in the framework of Nitsche's method and inf-sup stable velocity pressure pairs. The analysis was extended by Toselli to the *hp*-framework using mixed or equal order stabilized formulations in [14]. Local discontinuous Galerkin methods with equal order velocity and pressure spaces stabilized using penalty on the interelement pressure jumps was proposed by Cockburn et al. [8]. The Navier-Stokes equations discretized using DG has recently been given a full analysis in the framework of domain decomposition on non-matching meshes using DG techniques by Girault et al. [10].

In this paper we extend our previous work on low order discontinuous Galerkin methods for scalar second order elliptic problems to the case of incompressible flow problems [4, 5]. Using piecewise affine discontinuous finite elements enriched with non-conforming bubbles we may eliminate all stabilization terms from the formulation without compromising neither stability nor adjoint consistency. The upshot is that a local conservation property holds for the momentum equation independent of numerical parameters and optimal convergence in the L^2-norm may be proven using a duality argument. This is a consequence of the fact that the vectors of the velocity gradient matrix are functions in the lowest order Raviart-Thomas space for our choice of velocity finite element space (see also [3]).

Several choices for the pressure space are possible without introducing a penalty term. Indeed we can use either globally continuous, piecewise affine functions, piecewise constant, discontinuous functions or functions from a direct sum of these two sets of functions and still satisfy the inf-sup condition uniformly with respect to the mesh size.

Depending on the choice of pressure space slightly different results may be obtained. When the pressure space consists of continuous functions we prove that the

---

Date: June 10, 2009.

Key words and phrases. discontinuous Galerkin methods, Stokes' problem, bubble stabilization.

normal stresses are continuous, for piecewise constant data. Using spaces with discontinuous functions for the pressure on the other hand leads to a method that also enjoys local mass conservation. We give a unified analysis for these three choices of pressure space, with superconvergence estimates for certain residual quantities. We then discuss the differences of these approaches from numerical and analytical point of view.

2. NOTATION

Let Ω be a convex polygon (polyhedron in three space dimensions) in $\mathbb{R}^d$, $d = 2, 3$, with outer normal $\boldsymbol{n}$. Let $\mathcal{K}$ be a subdivision of $\Omega \subset \mathbb{R}^d$ into non-overlapping d-simplices κ and denote by $N_{\mathcal{K}}$ the number of simplices of the mesh. Suppose that each $\kappa \in \mathcal{K}$ is an affine image of the reference element $\widehat{\kappa}$, i.e. for each element κ there exists an affine transformation $T_\kappa : \widehat{\kappa} \to \kappa$.

Let $\mathcal{F}_i$ denote the set of interior faces ($(d-1)$-manifolds) of the mesh, i.e. the set of faces that are not included in the boundary $\partial\Omega$. The set $\mathcal{F}_e$ denotes the faces that are included in $\partial\Omega$ and define $\mathcal{F} = \mathcal{F}_i \cup \mathcal{F}_e$. Define by $N_{\mathcal{F}} = \operatorname{card}(\mathcal{F})$ and $N_{\mathcal{F}_i} = \operatorname{card}(\mathcal{F}_i)$ the number of faces resp. interior faces of the mesh.

Assume that $\mathcal{K}$ is shape-regular, does not contain any hanging node and covers $\overline{\Omega}$ exactly. For an element $\kappa \in \mathcal{K}$, h_κ denotes its diameter and for a face $F \in \mathcal{F}$, h_F denotes the diameter of F. Set $h = \max_{\kappa\in\mathcal{K}} h_\kappa$ and let $\tilde{h}$ be the function such that $\tilde{h}|_{\overset{\circ}{\kappa}} = h_\kappa$ and $\tilde{h}|_{\overset{\circ}{F}} = h_F$ for all $\kappa \in \mathcal{K}$ and $F \in \mathcal{F}$.

For a subset $R \subset \Omega$ or $R \subset \mathcal{F}$, $(\cdot,\cdot)_R$ denotes the $L^2(R)$–scalar product, $\|\cdot\|_R = (\cdot,\cdot)_R^{1/2}$ the corresponding norm, and $\|\cdot\|_{s,R}$ the $H^s(R)$–norm. The element-wise counterparts will be distinguished using the discrete partition as subscript, for example $(\cdot,\cdot)_{\mathcal{K}} = \sum_{\kappa\in\mathcal{K}} (\cdot,\cdot)_K$. For $s \geq 1$, let $H^s(\mathcal{K})$ be the space of piecewise Sobolev H^s–functions and denote its norm by $\|\cdot\|_{s,\mathcal{K}}$. Further let us denote $\boldsymbol{H}^s(\Omega) = [H^s(\Omega)]^d$ and $\boldsymbol{H}^s(\mathcal{K}) = [H^s(\mathcal{K})]^d$. Moreover the following space is defined

$$L^2_0(\Omega) = \left\{ v \in L^2(\Omega) \,\middle|\, \int_\Omega v \, dx = 0 \right\}.$$

Let $\boldsymbol{v} = (v_1, \ldots, v_d)^\top \in \boldsymbol{H}^1(\mathcal{K})$, then we define $\nabla \boldsymbol{v}|_\kappa \in [L^2(\kappa)]^{d\times d}$ by $(\nabla \boldsymbol{v})_{i,j} = \partial_{x_j} v_i$, $1 \leq i, j \leq d$, for each $\kappa \in \mathcal{K}$. Based on the scalar L^2-product we define

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{K}} = \sum_{i=1}^d (v_i, w_i)_{\mathcal{K}} \qquad \text{and} \qquad (\nabla \boldsymbol{v}, \nabla \boldsymbol{w})_{\mathcal{K}} = \sum_{i,j=1}^d (\partial_{x_j} v_i, \partial_{x_j} w_i)_{\mathcal{K}}$$

for all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^1(\mathcal{K})$.

Further let us define the jump and average operators. Fix $F \in \mathcal{F}_i$ and thus $F = \kappa_1 \cap \kappa_2$ with $\kappa_1, \kappa_2 \in \mathcal{K}$. Let $v \in H^1(\mathcal{K})$, $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{K})$ and denote by v_1, v_2 resp. $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ the restriction of v, $\boldsymbol{v}$ to the element κ_1, κ_2, i.e. $v_i = v|_{\kappa_i}$ resp. $\boldsymbol{v}_i = \boldsymbol{v}|_{\kappa_i}$, $i = 1, 2$, and denote by $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ the exterior normal of κ_1 resp. κ_2. We then define the average and jump operators by

$$\begin{aligned} \{v\} &= \tfrac{1}{2}(v_1 + v_2), & [v] &= v_1 \boldsymbol{n}_1 + v_2 \boldsymbol{n}_2, \\ \{\boldsymbol{v}\} &= \tfrac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2), & [\boldsymbol{v}] &= \boldsymbol{v}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{n}_2, \\ \{\nabla \boldsymbol{v}\} &= \tfrac{1}{2}(\nabla \boldsymbol{v}_1 + \nabla \boldsymbol{v}_2), & [\![\boldsymbol{v}]\!] &= \boldsymbol{v}_1 \otimes \boldsymbol{n}_1 + \boldsymbol{v}_2 \otimes \boldsymbol{n}_2, \\ & & [\![\nabla \boldsymbol{v}]\!] &= \nabla \boldsymbol{v}_1 \boldsymbol{n}_1 + \nabla \boldsymbol{v}_2 \boldsymbol{n}_2. \end{aligned}$$

where $\boldsymbol{a}\otimes\boldsymbol{b}\in\mathbb{R}^{d\times d}$ is defined by $(\boldsymbol{a}\otimes\boldsymbol{b})_{i,j}=a_ib_j$, $1\le i,j\le d$, for all $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^d$. Let $\boldsymbol{n}_{\mathtt{F}}\in\{\boldsymbol{n}_1,\boldsymbol{n}_2\}$ be arbitrarily chosen but fixed and observe that

$$\llbracket\boldsymbol{v}\rrbracket:\{\nabla\boldsymbol{w}\}=\llbracket\boldsymbol{v}\rrbracket\boldsymbol{n}_{\mathtt{F}}\cdot\{\nabla\boldsymbol{w}\}\boldsymbol{n}_{\mathtt{F}} \tag{1}$$

for all $\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{H}^1(\mathcal{K})$ and where $a:b=\sum_{i,j=1}^d a_{ij}b_{ij}$ for all $a,b\in\mathbb{R}^{d\times d}$. Additionally note that

$$\llbracket\boldsymbol{v}\rrbracket=\llbracket\boldsymbol{v}\rrbracket\boldsymbol{n}_{\mathtt{F}}\otimes\boldsymbol{n}_{\mathtt{F}},\qquad[\boldsymbol{v}]=\llbracket\boldsymbol{v}\rrbracket\boldsymbol{n}_{\mathtt{F}}\cdot\boldsymbol{n}_{\mathtt{F}}$$

and thus

$$\|\llbracket\boldsymbol{v}\rrbracket\|_{\mathcal{F}}=\|\llbracket\boldsymbol{v}\rrbracket\boldsymbol{n}_{\mathtt{F}}\|_{\mathcal{F}},\qquad\|[\boldsymbol{v}]\|_{\mathcal{F}}\le c\,\|\llbracket\boldsymbol{v}\rrbracket\|_{\mathcal{F}} \tag{2}$$

for some mesh independent constant $c>0$. On outer faces $F\in\mathcal{F}_e$ we define the jump and average operators by

$$\{v\}=v,\qquad\{\boldsymbol{v}\}=\boldsymbol{v},\qquad\{\nabla\boldsymbol{v}\}=\nabla\boldsymbol{v},$$

$$[v]=v\boldsymbol{n},\qquad[\boldsymbol{v}]=\boldsymbol{v}\cdot\boldsymbol{n},\qquad\llbracket\boldsymbol{v}\rrbracket=\boldsymbol{v}\otimes\boldsymbol{n},\qquad\llbracket\nabla\boldsymbol{v}\rrbracket=\nabla\boldsymbol{v}\boldsymbol{n},$$

where $\boldsymbol{n}$ is the outer normal of the domain Ω.

Lemma 2.1 (Integration by parts)**.** *Let* $v\in H^1(\mathcal{K})$ *and* $\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{H}^1(\mathcal{K})$*, then*

$$\begin{aligned}
(v,\nabla\cdot\boldsymbol{w})_{\mathcal{K}}&=-(\nabla v,\boldsymbol{w})_{\mathcal{K}}+(\{v\},[\boldsymbol{w}])_{\mathcal{F}}+([v],\{\boldsymbol{w}\})_{\mathcal{F}_i},\\
(\nabla\boldsymbol{v},\nabla\boldsymbol{w})_{\mathcal{K}}&=-(\Delta\boldsymbol{v},\boldsymbol{w})_{\mathcal{K}}+(\{\nabla\boldsymbol{v}\},\llbracket\boldsymbol{w}\rrbracket)_{\mathcal{F}}+(\llbracket\nabla\boldsymbol{v}\rrbracket,\{\boldsymbol{w}\})_{\mathcal{F}_i},\\
(\nabla\boldsymbol{v},\nabla\boldsymbol{w})_{\mathcal{K}}&=-(\Delta\boldsymbol{v},\boldsymbol{w})_{\mathcal{K}}+(\{\nabla\boldsymbol{v}\}\boldsymbol{n}_{\mathtt{F}},\llbracket\boldsymbol{w}\rrbracket\boldsymbol{n}_{\mathtt{F}})_{\mathcal{F}}+(\llbracket\nabla\boldsymbol{v}\rrbracket,\{\boldsymbol{w}\})_{\mathcal{F}_i}.
\end{aligned}$$

Proof. Element-wise integration by parts and applying the definitions of the jump and average operators leads to the first two equations. The third equation is deduced from the second by applying (1). □

3. Bubble stabilized finite element space

Let us denote by V_h^p the standard discontinuous finite element space of degree $p\ge0$ defined by

$$V_h^p=\left\{v_h\in L^2(\Omega)\;\middle|\;v_h|_\kappa\in\mathbb{P}_p(\kappa),\ \forall\kappa\in\mathcal{K}\right\},$$

where $\mathbb{P}_p(\kappa)$ denotes the set of polynomials of maximum degree p on κ. Consider then the enriched finite element space

$$V_{bs}=V_h^1\oplus\left\{v_h\in L^2(\Omega)\;\middle|\;v_h(x)=\alpha\,\boldsymbol{x}\cdot\boldsymbol{x},\ \alpha\in V_h^0\right\},$$

where $\boldsymbol{x}=(x_1,\dots,x_d)$ denotes the physical variables. Let us additionally define some functional spaces that consist of functions only defined on the skeleton of the mesh:

$$W_h^0=\left\{v_h\in L^2(\mathcal{F})\;\middle|\;v_h|_F\in\mathbb{P}_0(F),\ \forall F\in\mathcal{F}\right\}.$$

Define also the vectorial versions $\boldsymbol{V}_h^p=[V_h^p]^d$, $\boldsymbol{V}_{bs}=[V_{bs}]^d$ and $\boldsymbol{W}_h^0=[W_h^0]^d$.

Let $\boldsymbol{v}\in[L^2(\mathcal{F})]^m$, $m\in\{1,d,d^2\}$, and define by $\bar{\boldsymbol{v}}$ the L^2-projection of $\boldsymbol{v}$ onto $[W_h^0]^m$, i.e.

$$(\bar{\boldsymbol{v}},\boldsymbol{w}_h)_{\mathcal{F}}=(\boldsymbol{v},\boldsymbol{w}_h)_{\mathcal{F}},\qquad\forall\boldsymbol{w}_h\in[W_h^0]^m.$$

3.1. **Properties of the bubble stabilized finite element space.** Let us discuss some important properties of the space $\boldsymbol{V}_{bs}$.

Lemma 3.1. *For* $\boldsymbol{v}_h \in \boldsymbol{V}_{bs}$ *we have that*

$$\Delta \boldsymbol{v}_h \in \boldsymbol{V}_h^0.$$

Moreover the application $\Delta : \boldsymbol{V}_{bs}/V_h^1 \to \boldsymbol{V}_h^0$ *is bijective.*

Proof. Observe that $\Delta \boldsymbol{w}_h = 0$ for all $\boldsymbol{w}_h \in \boldsymbol{V}_h^1$ and that $\Delta(\boldsymbol{\alpha}\, \boldsymbol{x} \cdot \boldsymbol{x}) = 2d\boldsymbol{\alpha} \in \boldsymbol{V}_h^0$ where d is the dimension of Ω. ☐

Let us denote by $\boldsymbol{RT}_0(\kappa)$ the local lowest order Raviart-Thomas space on κ. The following Lemma holds.

Lemma 3.2. *For all* $\boldsymbol{v}_h \in \boldsymbol{V}_{bs}$ *with* $\boldsymbol{v}_h = (v_{h,1}, \dots, v_{h,d})^\top$ *there holds*

$$\nabla v_{h,i}|_\kappa \in \boldsymbol{RT}_0(\kappa), \quad \forall \kappa \in \mathcal{K},$$

and for all $\kappa \in \mathcal{K}$ *and* $\boldsymbol{r}_h = (\boldsymbol{r}_{h,1}, \dots, \boldsymbol{r}_{h,d})$ *with* $\boldsymbol{r}_{h,i} \in \boldsymbol{RT}_0(\kappa)$*, there exists* $\boldsymbol{v}_h \in V_{bs}$ *such that* $\nabla \boldsymbol{v}_{h,i}|_\kappa = \boldsymbol{r}_{h,i}$ *for all* $1 \le i \le d$.

Proof. For the scalar case we refer to the proof of Lemma 3.4 in [4] and the vectorial version is constructed component wise. ☐

Corollary 3.3. *For* $\boldsymbol{v}_h \in \boldsymbol{V}_{bs}$ *we have that*

$$\{\nabla \boldsymbol{v}_h\}\boldsymbol{n}_{\mathrm{F}} \in \boldsymbol{W}_h^0, \quad [\![\nabla \boldsymbol{v}_h]\!] \in \boldsymbol{W}_h^0.$$

Moreover the applications $\{\nabla \cdot\}\boldsymbol{n}_{\mathrm{F}} : \boldsymbol{V}_{bs} \to \boldsymbol{W}_h^0$ *and* $[\![\nabla \cdot]\!] : \boldsymbol{V}_{bs} \to \boldsymbol{W}_h^0$ *are surjective.*

Lemma 3.4. *There is a constant* $c > 0$ *independent of* h *such that for all* $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{K})$ *there holds*

$$\|\tilde{h}^{-\frac{1}{2}}[\![\boldsymbol{v}]\!]\|_{\mathcal{F}}^2 \le c\left(\|\tilde{h}^{-\frac{1}{2}}\overline{[\![\boldsymbol{v}]\!]}\|_{\mathcal{F}}^2 + \|\nabla \boldsymbol{v}\|_{\mathcal{K}}^2\right).$$

Proof. We refer to the proof of Lemma 4.1 in [2] for the scalar case. Its vectorial counterpart follows from the component wise construction. ☐

Lemma 3.5 (Poincaré inequality)**.** *There is a constant* $c > 0$ *independent of* h *such that for all* $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{K})$ *there holds*

$$\|\boldsymbol{v}\|_{\mathcal{K}}^2 \le c_P\left(\|\tilde{h}^{-\frac{1}{2}}\overline{[\![\boldsymbol{v}]\!]}\|_{\mathcal{F}}^2 + \|\nabla \boldsymbol{v}\|_{\mathcal{K}}^2\right).$$

Proof. This is a direct consequence of the previous Lemma and the Poincaré inequality proven by Brenner [1]. ☐

We define the standard norm for $\boldsymbol{H}^1(\mathcal{K})$-functions by

$$|||\boldsymbol{v}|||^2 = \|\nabla \boldsymbol{v}\|_{\mathcal{K}}^2 + \|\tilde{h}^{-\frac{1}{2}}[\![\boldsymbol{v}]\!]\|_{\mathcal{F}}^2$$

for all $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{K})$. Observe that there exists a constant $c_d > 0$ such that

$$c_d|||\boldsymbol{v}|||^2 \le \|\nabla \boldsymbol{v}\|_{\mathcal{K}}^2 + \|\tilde{h}^{-\frac{1}{2}}\overline{[\![\boldsymbol{v}]\!]}\|_{\mathcal{F}}^2 \le |||\boldsymbol{v}|||^2 \tag{3}$$

for all $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{K})$.

3.2. **Technical lemmas.** Let us cite some well known results. For the proofs we refer to [6].

Lemma 3.6 (Inverse inequality)**.** *Let* $\boldsymbol{v}_h \in [V_{bs}]^m$*,* $m \in \{1, d\}$*, then there exists a constant* $c_I > 0$ *independent of* $\tilde{h}$ *such that*

$$c_I^{-1}\|\tilde{h}^2 \Delta \boldsymbol{v}_h\|_{\mathcal{K}} \le \|\tilde{h}\nabla \boldsymbol{v}_h\|_{\mathcal{K}} \le c_I \|\boldsymbol{v}_h\|_{\mathcal{K}}.$$

Lemma 3.7 (Trace inequality)**.** *Let* $\boldsymbol{v} \in [H^1(\mathcal{K})]^{m_1}$ *and* $\boldsymbol{v}_h \in [V_{bs}]^{m_1}$*,* $m_1 \in \{1, d, d^2\}$ *resp.* $\boldsymbol{w} \in [H^1(\mathcal{K})]^d$ *and* $\boldsymbol{w}_h \in [V_{bs}]^d$*, then there holds*

$$\begin{aligned} \|\{\boldsymbol{v}\}\|_{\mathcal{F}} + \|[\boldsymbol{v}]\|_{\mathcal{F}} &\le c_T \left(\|\tilde{h}^{-\frac{1}{2}}\boldsymbol{v}\|_{\mathcal{K}} + \|\tilde{h}^{\frac{1}{2}}\nabla \boldsymbol{v}\|_{\mathcal{K}} \right), \\ \|\{\boldsymbol{v}_h\}\|_{\mathcal{F}} + \|[\boldsymbol{v}_h]\|_{\mathcal{F}} &\le c_T \|\tilde{h}^{-\frac{1}{2}}\boldsymbol{v}_h\|_{\mathcal{K}} \end{aligned}$$

and

$$\begin{aligned} \|[\![\boldsymbol{w}]\!]\|_{\mathcal{F}} &\le c_T \left(\|\tilde{h}^{-\frac{1}{2}}\boldsymbol{w}\|_{\mathcal{K}} + \|\tilde{h}^{\frac{1}{2}}\nabla \boldsymbol{w}\|_{\mathcal{K}} \right), \\ \|[\![\boldsymbol{w}_h]\!]\|_{\mathcal{F}} &\le c_T \|\tilde{h}^{-\frac{1}{2}}\boldsymbol{w}_h\|_{\mathcal{K}} \end{aligned}$$

where $c_T > 0$ *is a constant independent of* $\tilde{h}$*.*

3.3. **Projections.** Denote by $\pi_p : L^2(\Omega) \to V_h^p$ the L^2-projection onto V_h^p defined by

$$\int_\Omega \pi_k v \, w_h \, dx = \int_\Omega v \, w_h \, dx \qquad \forall w_h \in V_h^p$$

and by $\boldsymbol{\pi}_p : \boldsymbol{L}^2(\Omega) \to \boldsymbol{V}_h^p$ its vectorial counterpart defined by $(\boldsymbol{\pi}_p \boldsymbol{v})_i = \pi_p v_i$, $1 \le i \le d$. Then π_p and $\boldsymbol{\pi}_p$ satisfy the following approximation result: Let $v \in H^{p+1}(\mathcal{K})$ resp. $\boldsymbol{v} \in \boldsymbol{H}^{p+1}(\mathcal{K})$, then

$$\|v - \pi_p v\|_{\mathcal{K}} + \|\tilde{h}\nabla(v - \pi_p v)\|_{\mathcal{K}} \le c h^{p+1} |v|_{p+1,\mathcal{K}}, \tag{4}$$

$$\|\boldsymbol{v} - \boldsymbol{\pi}_p \boldsymbol{v}\|_{\mathcal{K}} + \|\tilde{h}\nabla(\boldsymbol{v} - \boldsymbol{\pi}_p \boldsymbol{v})\|_{\mathcal{K}} \le c h^{p+1} |\boldsymbol{v}|_{p+1,\mathcal{K}}. \tag{5}$$

Denote the Crouzeix-Raviart space by CR and its vectorial counterpart by $\boldsymbol{CR}$. Additionally let us denote by $\boldsymbol{i}_c : \boldsymbol{H}^1(\Omega) \to \boldsymbol{CR}$ the vectorial Crouzeix-Raviart interpolant satisfying the following approximation result: if $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega)$, then

$$\|\boldsymbol{v} - \boldsymbol{i}_c \boldsymbol{v}\|_{\mathcal{K}} + \|\tilde{h}\nabla(\boldsymbol{v} - \boldsymbol{i}_c \boldsymbol{v})\|_{\mathcal{K}} \le c h^2 |\boldsymbol{v}|_{2,\mathcal{K}}. \tag{6}$$

Further denote by $C_h : L^2(\Omega) \to V_{h,c}^1$ the Clément interpolant [7] where $V_{h,c}^1 = \left\{ v_h \in C^0(\overline{\Omega}) \,\middle|\, v_h|_\kappa \in \mathbb{P}_1(\kappa),\ \forall \kappa \in \mathcal{K} \right\}$ is the continuous finite element space of degree 1. Recall that the Clément interpolant satisfies

$$\|v - C_h v\|_{\mathcal{K}} + \|\tilde{h}\nabla(v - C_h v)\|_{\mathcal{K}} \le c h^{\gamma+1} |v|_{\gamma+1,\mathcal{K}} \tag{7}$$

for all $v \in H^{\gamma+1}(\mathcal{K})$, $\gamma \in \{0, 1\}$. Note that one can prove that the Clément interpolant conserves the mean of a function over Ω, i.e., $\int_\Omega C_h v(x)\, dx = \int_\Omega v(x)\, dx$, and thus $C_h(L_0^2(\Omega)) \subset L_0^2(\Omega)$. Note also that C_h is H^1-stable, i.e. $\|\nabla(C_h v)\|_{\mathcal{K}} \le c\, \|\nabla v\|_{\mathcal{K}}$. We denote by $\boldsymbol{C}_h$ the vectorial version of C_h sharing all properties.

Furthermore, we present the following projection which will be used in the analysis.

Lemma 3.8. *Let* $\boldsymbol{a}_h \in \boldsymbol{V}_h^0$ *and* $\boldsymbol{b}_h, \boldsymbol{c}_h \in \boldsymbol{W}_h^0$ *be fixed. Then, there exists a unique function* $\boldsymbol{\phi}_h \in \boldsymbol{V}_{bs}$ *such that*

$$\left\{\begin{aligned} \boldsymbol{\pi}_0 \boldsymbol{\phi}_h &= \boldsymbol{a}_h, \\ \{\nabla \boldsymbol{\phi}_h\}|_F \boldsymbol{n}_{\mathrm{F}} &= \boldsymbol{b}_h|_F \quad \forall F \in \mathcal{F}, \\ \overline{\{\boldsymbol{\phi}_h\}}|_F &= \boldsymbol{c}_h|_F \quad \forall F \in \mathcal{F}_i. \end{aligned}\right. \tag{8}$$

Moreover $\boldsymbol{\phi}_h$ *satisfies the following stability result*

$$\|\tilde{h}^{-1}\boldsymbol{\phi}_h\|_{\mathcal{K}}^2 + |||\boldsymbol{\phi}_h|||^2 \le c\left(\|\tilde{h}^{-1}\boldsymbol{a}_h\|_{\mathcal{K}}^2 + \|\tilde{h}^{\frac{1}{2}}\boldsymbol{b}_h\|_{\mathcal{F}}^2 + \|\tilde{h}^{-\frac{1}{2}}\boldsymbol{c}_h\|_{\mathcal{F}_i}^2\right). \tag{9}$$

Proof. Let us first establish the a priori estimate. Firstly by the trace inequality observe that

$$\|\tilde{h}^{-\frac{1}{2}}\llbracket\boldsymbol{\phi}_h\rrbracket\|_{\mathcal{F}}^2 \le c\,\|\tilde{h}^{-1}\boldsymbol{\phi}_h\|_{\mathcal{K}}^2 \tag{10}$$

and using (5) that

$$\|\tilde{h}^{-1}\boldsymbol{\phi}_h\|_{\mathcal{K}}^2 \le \|\tilde{h}^{-1}\pi_0\boldsymbol{\phi}_h\|_{\mathcal{K}}^2 + \|\tilde{h}^{-1}(\boldsymbol{\phi}_h - \pi_0\boldsymbol{\phi}_h)\|_{\mathcal{K}}^2 \le \|\tilde{h}^{-1}\boldsymbol{a}_h\|_{\mathcal{K}}^2 + c_\star\,\|\nabla\boldsymbol{\phi}_h\|_{\mathcal{K}}^2 \tag{11}$$

for some constant $c_\star > 0$. Secondly, use Lemma 2.1, 3.1 and Corollary 3.3

$$\begin{aligned} \|\nabla\boldsymbol{\phi}_h\|_{\mathcal{K}}^2 &= -(\Delta\boldsymbol{\phi}_h, \pi_0\boldsymbol{\phi}_h)_{\mathcal{K}} + (\{\nabla\boldsymbol{\phi}_h\}\boldsymbol{n}_{\mathrm{F}}, \overline{\llbracket\boldsymbol{\phi}_h\rrbracket}\boldsymbol{n}_{\mathrm{F}})_{\mathcal{F}} + (\llbracket\nabla\boldsymbol{\phi}_h\rrbracket, \overline{\{\boldsymbol{\phi}_h\}})_{\mathcal{F}_i} \\ &= \underbrace{-(\Delta\boldsymbol{\phi}_h, \boldsymbol{a}_h)_{\mathcal{K}}}_{I} + \underbrace{(\overline{\llbracket\boldsymbol{\phi}_h\rrbracket}\boldsymbol{n}_{\mathrm{F}}, \boldsymbol{b}_h)_{\mathcal{F}}}_{II} + \underbrace{(\llbracket\nabla\boldsymbol{\phi}_h\rrbracket, \boldsymbol{c}_h)_{\mathcal{F}_i}}_{III}. \end{aligned}$$

Applying (11) and the Cauchy-Schwarz, the inverse (Lemma 3.6), the trace (Lemma 3.7) and Young's inequalities for each term yields respectively

$$\begin{aligned} I &\le c_I\|\nabla\boldsymbol{\phi}_h\|_{\mathcal{K}}\|\tilde{h}^{-1}\boldsymbol{a}_h\|_{\mathcal{K}} \le \tfrac{1}{4}\|\nabla\boldsymbol{\phi}_h\|_{\mathcal{K}}^2 + c_I^2\|\tilde{h}^{-1}\boldsymbol{a}_h\|_{\mathcal{K}}^2 \\ II &\le c_T\|\tilde{h}^{-1}\boldsymbol{\phi}_h\|_{\mathcal{K}}\|\tilde{h}^{\frac{1}{2}}\boldsymbol{b}_h\|_{\mathcal{F}} \le \tfrac{1}{4}\|\nabla\boldsymbol{\phi}_h\|_{\mathcal{K}}^2 + \tfrac{1}{4c_\star}\|\tilde{h}^{-1}\boldsymbol{a}_h\|_{\mathcal{K}}^2 + c_\star c_T^2\|\tilde{h}^{\frac{1}{2}}\boldsymbol{b}_h\|_{\mathcal{F}}^2, \\ III &\le c_T\|\nabla\boldsymbol{\phi}_h\|_{\mathcal{K}}\|\tilde{h}^{-\frac{1}{2}}\boldsymbol{c}_h\|_{\mathcal{F}_i} \le \tfrac{1}{4}\|\nabla\boldsymbol{\phi}_h\|_{\mathcal{K}}^2 + c_T^2\|\tilde{h}^{-\frac{1}{2}}\boldsymbol{c}_h\|_{\mathcal{F}_i}^2, \end{aligned}$$

and thus

$$\|\nabla\boldsymbol{\phi}_h\|_{\mathcal{K}}^2 \le c\left(\|\tilde{h}^{-1}\boldsymbol{a}_h\|_{\mathcal{K}}^2 + \|\tilde{h}^{\frac{1}{2}}\boldsymbol{b}_h\|_{\mathcal{F}}^2 + \|\tilde{h}^{-\frac{1}{2}}\boldsymbol{c}_h\|_{\mathcal{F}_i}^2\right),$$

which, combined with (3), (10) and (11), completes the a priori estimate. To conclude the proof, it now suffices to observe that (8) is a square linear system of size $d\,(N_{\mathcal{K}} + N_{\mathcal{F}} + N_{\mathcal{F}_i})$. Hence, existence and uniqueness of a solution of the linear system are equivalent. Let us denote by $Aw = b$ the square linear system and assume that there is a vector w_1 and w_2 such that $Aw_i = b$, $i = 1, 2$. Further let us denote the difference between them by $e = w_1 - w_2$ and therefore $Ae = 0$. The a priori estimate (9) implies that $e = 0$ and thus the solution is unique and hence the matrix is regular. □

4. Bubble stabilized discontinuous Galerkin method for Stokes' problem

Consider the steady Stokes problem: find $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that

$$\left\{\begin{aligned} -\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \partial\Omega, \end{aligned}\right. \tag{12}$$

where $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\partial\Omega)$ such that $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n}\, ds = 0$. This setting ensures a unique solution to the model problem (12), see [9].

4.1. **Bubble stabilized discontinuous Galerkin method.**

Let $Q_h \subset L^2_0(\Omega)$ be some scalar finite element space that will be specified later. We introduce the bubble stabilized Galerkin method by: find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{bs} \times Q_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(p_h, \boldsymbol{v}_h) - b(q_h, \boldsymbol{u}_h) = F(\boldsymbol{v}_h, q_h) \quad \forall (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{bs} \times Q_h \tag{13}$$

where the linear form $F(\cdot,\cdot)$ and the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}_h) &= (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h)_{\mathcal{K}} - (\{\nabla \boldsymbol{u}_h\}, [\![\boldsymbol{v}_h]\!])_{\mathcal{F}} - ([\![\boldsymbol{u}_h]\!], \{\nabla \boldsymbol{v}_h\})_{\mathcal{F}}, \\ b(p_h, \boldsymbol{v}_h) &= -(p_h, \nabla \cdot \boldsymbol{v}_h)_{\mathcal{K}} + (\{p_h\}, [\boldsymbol{v}_h])_{\mathcal{F}}, \\ F(\boldsymbol{v}_h, q_h) &= (\boldsymbol{f}, \boldsymbol{v}_h)_{\mathcal{K}} - (\boldsymbol{g} \cdot \boldsymbol{n}, q_h)_{\mathcal{F}_e} - (\boldsymbol{g}, \nabla \boldsymbol{v}_h \boldsymbol{n})_{\mathcal{F}_e}. \end{aligned}$$

Remark 4.1. The discrete solution $\boldsymbol{u}_h$ and p_h of (13) satisfies the following local linear momentum conservation property

$$-\int_{\partial\kappa} \{\nabla \boldsymbol{u}_h\} \boldsymbol{n}_\kappa \, ds + \int_{\partial\kappa} \{p_h\} \boldsymbol{n}_\kappa \, ds = \int_\kappa \boldsymbol{f} \, dx$$

for all $\kappa \in \mathcal{K}$ and where $\boldsymbol{n}_\kappa$ denotes the outer normal of κ.

Lemma 4.2 (Consistency)**.** *Let the pair* $(\boldsymbol{u}, p) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ *be the exact solution of* (12) *and let* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{bs} \times Q_h$ *be the approximation defined by* (13)*. Then, there holds that*

$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + b(p - p_h, \boldsymbol{v}_h) - b(q_h, \boldsymbol{u} - \boldsymbol{u}_h) = 0$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{bs} \times Q_h$.

Proof. By integration by parts, Lemma 2.1. Details are left to the reader. □

5. Convergence analysis

Assume for simplicity homogeneous boundary conditions, i.e. $\boldsymbol{g} = \boldsymbol{0}$, in this section.

We will specify the choice for the pressure space Q_h to get inf-sup stable approximations. Let us first introduce three possible choices for Q_h. The first alternative to define the pressure approximation space is as the continuous piecewise linear finite element space defined by

$$Q^1_{h,c} = \left\{ v_h \in C^0(\overline{\Omega}) \cap L^2_0(\Omega) \,\middle|\, v_h|_\kappa \in \mathbb{P}_1(\kappa), \ \forall \kappa \in \mathcal{K} \right\}$$

and the second one is the space of piecewise constant functions

$$Q^0_{h,d} = \left\{ v_h \in L^2_0(\Omega) \,\middle|\, v_h|_\kappa \in \mathbb{P}_0(\kappa), \ \forall \kappa \in \mathcal{K} \right\}.$$

Further let us also consider the direct sum of the above defined spaces $Q^1_{h,c} \oplus Q^0_{h,d}$. Remark that $\|\tilde{h} \nabla q_h\|_{\mathcal{K}}$ and $\|\tilde{h}^{\frac{1}{2}} [q_h]\|_{\mathcal{F}_i}$ is a norm for $q_h \in Q^1_{h,c}$ resp. $q_h \in Q^0_{h,d}$ and thus

$$|||q_h|||^2_q = \|\tilde{h} \nabla q_h\|^2_{\mathcal{K}} + \|\tilde{h}^{\frac{1}{2}} [q_h]\|^2_{\mathcal{F}_i}$$

is a norm for $q_h \in Q^1_{h,c} \oplus Q^0_{h,d}$. For the following we chose $Q_h = Q^1_{h,c} \oplus Q^0_{h,d}$ and observe that the cases $Q_h = Q^1_{h,c}$ and $Q_h = Q^0_{h,d}$ are covered by the analysis as well.

Thus let us define the following triple norm

$$|||\boldsymbol{v}_h, q_h|||^2 = |||\boldsymbol{v}_h|||^2 + |||q_h|||^2_q$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{bs} \times Q_h$.

Proposition 5.1 (Inf-sup condition)**.** *There exists a constant $c > 0$ independent of h such that there holds $\forall (\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{bs} \times Q_h$*

$$c\, |||\boldsymbol{u}_h, p_h||| \leq \sup_{\boldsymbol{0} \neq (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{bs} \times Q_h} \frac{a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(p_h, \boldsymbol{v}_h) - b(q_h, \boldsymbol{u}_h)}{|||\boldsymbol{v}_h, q_h|||}.$$

Proof. For the proof of Proposition 5.1 we introduce the following two lemmas.

Lemma 5.2. *There exists a constant $c > 0$ independent of h such that for each fixed pair $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{bs} \times Q_h$ there exists a pair $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{bs} \times Q_h$ with*

$$c\, |||\boldsymbol{u}_h, p_h|||^2 \leq a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(p_h, \boldsymbol{v}_h) - b(q_h, \boldsymbol{u}_h).$$

Lemma 5.3. *There exists a constant $c > 0$ independent of h such that for each $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{bs} \times Q_h$ there exists a pair $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{bs} \times Q_h$ such that*

$$|||\boldsymbol{v}_h, q_h||| \leq c\, |||\boldsymbol{u}_h, p_h|||.$$

Indeed combining Lemma 5.2 and 5.3 yields

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(p_h, \boldsymbol{v}_h) - b(q_h, \boldsymbol{u}_h) \geq c\, |||\boldsymbol{u}_h, p_h|||^2 \geq c\, |||\boldsymbol{u}_h, p_h||| \, |||\boldsymbol{v}_h, q_h|||.$$

□

Proof of Lemma 5.2. Firstly fix $\boldsymbol{u}_h \in \boldsymbol{V}_{bs}$ and $p_h \in Q_h$. Observe that choosing $\boldsymbol{v}_h = \boldsymbol{u}_h$ and $q_h = p_h$ in (13) yields

$$\begin{aligned} & a(\boldsymbol{u}_h, \boldsymbol{u}_h) \\ & = \|\nabla \boldsymbol{u}_h\|_{\mathcal{K}}^2 - 2(\{\nabla \boldsymbol{u}_h\}, \overline{\llbracket \boldsymbol{u}_h \rrbracket})_{\mathcal{F}} \geq \|\nabla \boldsymbol{u}_h\|_{\mathcal{K}}^2 - 2\|\tilde{h}^{\frac{1}{2}} \{\nabla \boldsymbol{u}_h\}\|_{\mathcal{F}} \|\tilde{h}^{-\frac{1}{2}} \overline{\llbracket \boldsymbol{u}_h \rrbracket}\|_{\mathcal{F}} \\ & \geq \|\nabla \boldsymbol{u}_h\|_{\mathcal{K}}^2 - 2c_T \|\nabla \boldsymbol{u}_h\|_{\mathcal{K}} \|\tilde{h}^{-\frac{1}{2}} \overline{\llbracket \boldsymbol{u}_h \rrbracket}\|_{\mathcal{F}} \geq \tfrac{1}{2}\|\nabla \boldsymbol{u}_h\|_{\mathcal{K}}^2 - c_1 \|\tilde{h}^{-\frac{1}{2}} \overline{\llbracket \boldsymbol{u}_h \rrbracket}\|_{\mathcal{F}}^2 \end{aligned} \tag{14}$$

using Corollary 3.3, the Cauchy-Schwarz, the trace and Young's inequalities.

Further let $\boldsymbol{w}_h \in \boldsymbol{V}_{bs}$ be the projection defined by Lemma 3.8 with $\boldsymbol{a}_h = \boldsymbol{0}$, $\boldsymbol{b}_h = -\tilde{h}^{-1} \overline{\llbracket \boldsymbol{u}_h \rrbracket} \boldsymbol{n}_{\mathtt{F}}$ and $\boldsymbol{c}_h = \boldsymbol{0}$. By its properties, Lemma 2.1 and 3.1, Corollary 3.3 and (2) we get

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{w}_h) & = -(\Delta \boldsymbol{u}_h, \boldsymbol{w}_h)_{\mathcal{K}} + (\llbracket \nabla \boldsymbol{u}_h \rrbracket, \{\boldsymbol{w}_h\})_{\mathcal{F}_i} - (\llbracket \boldsymbol{u}_h \rrbracket \boldsymbol{n}_{\mathtt{F}}, \{\nabla \boldsymbol{w}_h\} \boldsymbol{n}_{\mathtt{F}})_{\mathcal{F}} \\ & = -(\Delta \boldsymbol{u}_h, \boldsymbol{\pi}_0 \boldsymbol{w}_h)_{\mathcal{K}} + (\llbracket \nabla \boldsymbol{u}_h \rrbracket, \overline{\{\boldsymbol{w}_h\}})_{\mathcal{F}_i} - (\overline{\llbracket \boldsymbol{u}_h \rrbracket} \boldsymbol{n}_{\mathtt{F}}, \{\nabla \boldsymbol{w}_h\} \boldsymbol{n}_{\mathtt{F}})_{\mathcal{F}} \\ & = \|\tilde{h}^{-\frac{1}{2}} \overline{\llbracket \boldsymbol{u}_h \rrbracket}\|_{\mathcal{F}}^2 \end{aligned} \tag{15}$$

and again by integration by parts

$$b(p_h, \boldsymbol{w}_h) = (\nabla p_h, \boldsymbol{\pi}_0 \boldsymbol{w}_h)_{\mathcal{K}} - ([p_h], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} = -([p_h], \{\boldsymbol{w}_h\})_{\mathcal{F}_i}$$

since $\nabla p_h \in \boldsymbol{V}_h^0$ and by the property of the projection $\boldsymbol{w}_h$. Further since $p_h \in Q_h$ we write $p_h = p_{h,c} + p_{h,d}$ with $p_{h,c} \in Q_{h,c}^1$ and $p_{h,d} \in Q_{h,d}^0$ and thus

$$b(p_h, \boldsymbol{w}_h) = -([p_{h,d}], \overline{\{\boldsymbol{w}_h\}})_{\mathcal{F}_i} = 0$$

since $[p_{h,c}] = 0$ and by the property of the projection $\boldsymbol{w}_h$.

Let $\boldsymbol{z}_h$ be the projection defined by Lemma 3.8 with $\boldsymbol{a}_h = \tilde{h}^2 \nabla p_h$, $\boldsymbol{b}_h = \boldsymbol{0}$ and $\boldsymbol{c}_h = -\tilde{h}[p_h]$ and observe that

$$b(p_h, \boldsymbol{z}_h) = (\nabla p_h, \boldsymbol{\pi}_0 \boldsymbol{z}_h)_{\mathcal{K}} - ([p_{h,d}], \overline{\{\boldsymbol{z}_h\}})_{\mathcal{F}_i} = \|\tilde{h} \nabla p_h\|_{\mathcal{K}}^2 + \|\tilde{h}^{\frac{1}{2}} [p_h]\|_{\mathcal{F}_i}^2 = |||p_h|||_q^2.$$

Also, applying integration by parts (Lemma 2.1) combined with the Cauchy-Schwarz and Young's inequality, yields

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{z}_h) &= -(\Delta \boldsymbol{u}_h, \boldsymbol{\pi}_0 \boldsymbol{z}_h)_{\mathcal{K}} + ([\![\nabla \boldsymbol{u}_h]\!], \overline{\{\boldsymbol{z}_h\}})_{\mathcal{F}_i} - (\overline{[\![\boldsymbol{u}_h]\!]} \boldsymbol{n}_{\mathtt{F}}, \{\nabla \boldsymbol{z}_h\} \boldsymbol{n}_{\mathtt{F}})_{\mathcal{F}} \\ &= -(\Delta \boldsymbol{u}_h, \tilde{h}^2 \nabla p_h)_{\mathcal{K}} - ([\![\nabla \boldsymbol{u}_h]\!], \tilde{h}[p_h])_{\mathcal{F}_i} \\ &\geq -c \, \|\nabla \boldsymbol{u}_h\|_{\mathcal{K}} \left(\|\tilde{h} \nabla p_h\|_{\mathcal{K}} + \|\tilde{h}^{\frac{1}{2}} [p_h]\|_{\mathcal{F}_i} \right) \geq -c \, \|\nabla \boldsymbol{u}_h\|_{\mathcal{K}}^2 - \tfrac{1}{2} |||p_h|||_q^2 \end{aligned}$$

which implies that

$$a(\boldsymbol{u}_h, \boldsymbol{z}_h) + b(p_h, \boldsymbol{z}_h) \geq \tfrac{1}{2} |||p_h|||_q^2 - c_2 \|\nabla \boldsymbol{u}_h\|_{\mathcal{K}}^2. \tag{16}$$

Defining $\boldsymbol{v}_h = \boldsymbol{u}_h + (c_1 + \frac{1}{2}) \boldsymbol{w}_h + \frac{1}{4c_2} \boldsymbol{z}_h$, $q_h = p_h$ and respecting (14)-(16) yields

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(p_h, \boldsymbol{v}_h) - b(q_h, \boldsymbol{u}_h) \geq c \, |||\boldsymbol{u}_h, p_h|||^2$$

using (3). □

Proof of Lemma 5.3. Observe that by (9) we get

$$|||\boldsymbol{w}_h|||^2 \leq c \, |||\boldsymbol{u}_h|||^2 \quad \text{and} \quad |||\boldsymbol{z}_h|||^2 \leq c \, |||p_h|||_q^2,$$

and therefore using the definition of $\boldsymbol{v}_h$ implies

$$\begin{aligned} |||\boldsymbol{v}_h, q_h|||^2 = |||\boldsymbol{v}_h|||^2 + |||p_h|||_q^2 &\leq 4 \, |||\boldsymbol{u}_h|||^2 + 4 \, |||\boldsymbol{w}_h|||^2 + 4 \, |||\boldsymbol{z}_h|||^2 + |||p_h|||_q^2 \\ &\leq c \, (|||\boldsymbol{u}_h|||^2 + |||p_h|||_q^2) = c \, |||\boldsymbol{u}_h, p_h|||^2. \end{aligned}$$

□

Thus the numerical scheme is inf-sup stable for all three choices $Q_h = Q_{h,c}^1$, $Q_h = Q_{h,d}^0$ and $Q_h = Q_{h,c}^1 \oplus Q_{h,d}^0$. Let us define the following auxiliary norm

$$|||\boldsymbol{v}, q|||_a^2 = \|\tilde{h}^{-1} \boldsymbol{v}\|_{\mathcal{K}}^2 + \|\nabla \boldsymbol{v}\|_{\mathcal{K}}^2 + \|\tilde{h}^{-\frac{1}{2}} \{\boldsymbol{v}\}\|_{\mathcal{F}_i}^2 + \|\tilde{h}^{\frac{1}{2}} \{\nabla \boldsymbol{v}\}\|_{\mathcal{F}}^2 + \|q\|_{\mathcal{K}}^2 + \|\tilde{h}^{\frac{1}{2}} \{q\}\|_{\mathcal{F}}^2$$

for all $\boldsymbol{v} \in \boldsymbol{H}^2(\mathcal{K})$ and $q \in L_0^2(\Omega)$ which will be used for the continuity result.

Proposition 5.4 (Continuity)**.** *Let $\boldsymbol{v} \in \boldsymbol{H}^2(\mathcal{K})$, $\boldsymbol{v}_h \in \boldsymbol{V}_{bs}$, $q \in H^1(\mathcal{K})$ and $q_h \in Q_h$. There exists a constant $c > 0$ independent of h such that*

$$a(\boldsymbol{v}, \boldsymbol{v}_h) + b(q, \boldsymbol{v}_h) - b(q_h, \boldsymbol{v}) \leq c \, |||\boldsymbol{v}, q|||_a |||\boldsymbol{v}_h, q_h|||.$$

Proof. Applying the Cauchy-Schwarz inequality, Lemma 3.4 and (2) yields

$$\begin{aligned}
a(\boldsymbol{v},\boldsymbol{v}_h) &= (\nabla\boldsymbol{v},\nabla\boldsymbol{v}_h)_{\mathcal{K}} - (\{\nabla\boldsymbol{v}\},\llbracket\boldsymbol{v}_h\rrbracket)_{\mathcal{F}}\\
&\le \left(\|\nabla\boldsymbol{v}\|_{\mathcal{K}}^2 + \|\tilde{h}^{\frac12}\{\nabla\boldsymbol{v}\}\|_{\mathcal{F}}^2\right)^{\frac12}\left(\|\nabla\boldsymbol{v}_h\|_{\mathcal{K}}^2 + \|\tilde{h}^{-\frac12}\llbracket\boldsymbol{v}_h\rrbracket\|_{\mathcal{F}}^2\right)^{\frac12}\\
&\le |||\boldsymbol{v},q|||_a|||\boldsymbol{v}_h,q_h|||,\\
b(q,\boldsymbol{v}_h) &= -(q,\nabla\cdot\boldsymbol{v}_h)_{\mathcal{K}} + (\{q\},[\boldsymbol{v}_h])_{\mathcal{F}}\\
&\le \left(\|q\|_{\mathcal{K}}^2 + \|\tilde{h}^{\frac12}\{q\}\|_{\mathcal{F}}^2\right)^{\frac12}\left(\|\nabla\boldsymbol{v}_h\|_{\mathcal{K}}^2 + \|\tilde{h}^{-\frac12}[\boldsymbol{v}_h]\|_{\mathcal{F}}^2\right)^{\frac12}\\
&\le \left(\|q\|_{\mathcal{K}}^2 + \|\tilde{h}^{\frac12}\{q\}\|_{\mathcal{F}}^2\right)^{\frac12}\left(\|\nabla\boldsymbol{v}_h\|_{\mathcal{K}}^2 + \|\tilde{h}^{-\frac12}\llbracket\boldsymbol{v}_h\rrbracket\|_{\mathcal{F}}^2\right)^{\frac12}\\
&\le |||\boldsymbol{v},q|||_a|||\boldsymbol{v}_h,q_h|||,\\
-b(q_h,\boldsymbol{v}) &= (\nabla q_h,\boldsymbol{v})_{\mathcal{K}} - ([q_h],\{\boldsymbol{v}\})_{\mathcal{F}_i}\\
&\le \left(\|\tilde{h}\nabla q_h\|_{\mathcal{K}}^2 + \|\tilde{h}^{\frac12}[q_h]\|_{\mathcal{F}_i}^2\right)^{\frac12}\left(\|\tilde{h}^{-1}\boldsymbol{v}\|_{\mathcal{K}}^2 + \|\tilde{h}^{-\frac12}\{\boldsymbol{v}\}\|_{\mathcal{F}_i}^2\right)^{\frac12}\\
&\le c\,|||\boldsymbol{v},q|||_a|||\boldsymbol{v}_h,q_h|||.
\end{aligned}$$

□

Assume that $Q_h = \alpha\, Q_{h,c}^1 \oplus \beta\, Q_{h,d}^0$ for $(\alpha,\beta)\in\{(1,0),(0,1),(1,1)\}$ and define in a standard manner the following quantities

$$\begin{aligned}
\boldsymbol{\eta_u} &= \boldsymbol{u} - \boldsymbol{i}_c\boldsymbol{u}, &&\qquad & \eta_p &= p - P_\alpha p,\\
\boldsymbol{\xi_u} &= \boldsymbol{u}_h - \boldsymbol{i}_c\boldsymbol{u}, &&\text{and} & \xi_p &= p_h - P_\alpha p,
\end{aligned}\tag{17}$$

where the projection $P_\alpha : L^2(\Omega)\to Q_h$ is defined by $P_\alpha = (1-\alpha)\,\pi_0 + \alpha\, C_h$. Observe that $\boldsymbol{\xi_u}\in\boldsymbol{V}_{bs}$ and $\xi_p\in Q_h$ since the projection P_α preserves the property of zero mean. Further P_α satisfies the error estimate

$$\|v - P_\alpha v\|_{\mathcal{K}} + \|\tilde{h}\nabla(v - P_\alpha v)\|_{\mathcal{K}} \le ch^{\alpha\gamma+1}|v|_{\alpha\gamma+1,\mathcal{K}}\tag{18}$$

for all $v\in H^{\gamma+1}(\mathcal{K})$, $\gamma\in\{0,1\}$. We denote further by $A+B$ the (in general not direct) sum of the functional spaces A and B. Then we prove the following continuity result.

Proposition 5.5 (Approximability)**.** *Let $\boldsymbol{\eta_u}\in\boldsymbol{H}^2(\Omega)+\boldsymbol{CR}$ and $\eta_p\in H^{\gamma+1}(\Omega)+Q_h$, with $\gamma\in\{0,1\}$ and $Q_h=\alpha\, Q_{h,c}^1\oplus\beta\, Q_{h,d}^0$ for $(\alpha,\beta)\in\{(1,0),(0,1),(1,1)\}$, be defined by* (17)*. Then, there exists a constant $c>0$ independent of h such that*

$$|||\boldsymbol{\eta_u},\eta_p||| + |||\boldsymbol{\eta_u},\eta_p|||_a \le ch|\boldsymbol{u}|_{2,\mathcal{K}} + ch^{1+\alpha\gamma}|p|_{1+\alpha\gamma,\mathcal{K}}.$$

Proof. This is a direct consequence of the trace inequality (Lemma 3.7) and the error estimates (6) and (18). □

Theorem 5.6 (Convergence in Energy norm)**.** *Let $\boldsymbol{u}\in\boldsymbol{H}^2(\Omega)$, $p\in H^{\gamma+1}(\Omega)$, $\gamma\in\{0,1\}$, be the exact solution of problem* (12) *and let $\boldsymbol{u}_h\in\boldsymbol{V}_{bs}$, $p_h\in Q_h$, with $Q_h=\alpha\, Q_{h,c}^1\oplus\beta\, Q_{h,d}^0$ for $(\alpha,\beta)\in\{(1,0),(0,1),(1,1)\}$, be the approximation defined by* (13)*. Then, there exists a constant $c>0$ independent of h such that*

$$|||\boldsymbol{u}-\boldsymbol{u}_h||| + \|p-p_h\|_{\mathcal{K}} + |||p-p_h|||_q \le ch|\boldsymbol{u}|_{2,\mathcal{K}} + ch^{1+\alpha\gamma}|p|_{1+\alpha\gamma,\mathcal{K}}.$$

Proof. Let us first establish the a priori estimate for the triple norm $|||\cdot|||$. Split the error in a standard manner in two parts

$$|||\boldsymbol{u}-\boldsymbol{u}_h, p-p_h||| \le |||\boldsymbol{\eta_u},\eta_p||| + |||\boldsymbol{\xi_u},\xi_p|||.\tag{19}$$

By Proposition 5.5 it follows that

$$|||\boldsymbol{\eta}_{\boldsymbol{u}}, \eta_p||| \le ch|\boldsymbol{u}|_{2,\mathcal{K}} + ch^{1+\alpha\gamma}|p|_{1+\alpha\gamma,\mathcal{K}}.$$

For the second term of the right hand side of (19) observe that applying the inf-sup condition of Proposition 5.1, the consistency of Lemma 4.2, the continuity of Proposition 5.4 and the approximability of Proposition 5.5 yields

$$\begin{aligned}
|||\boldsymbol{\xi}_{\boldsymbol{u}}, \xi_p||| &\le c \sup_{\boldsymbol{0}\neq(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_{bs}\times Q_h} \frac{a(\boldsymbol{\xi}_{\boldsymbol{u}},\boldsymbol{v}_h) + b(\xi_p,\boldsymbol{v}_h) - b(q_h,\boldsymbol{\xi}_{\boldsymbol{u}})}{|||\boldsymbol{v}_h, q_h|||} \\
&= c \sup_{\boldsymbol{0}\neq(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_{bs}\times Q_h} \frac{a(\boldsymbol{\eta}_{\boldsymbol{u}},\boldsymbol{v}_h) + b(\eta_p,\boldsymbol{v}_h) - b(q_h,\boldsymbol{\eta}_{\boldsymbol{u}})}{|||\boldsymbol{v}_h, q_h|||} \\
&\le c\,|||\boldsymbol{\eta}_{\boldsymbol{u}}, \eta_p|||_a \le ch|\boldsymbol{u}|_{2,\mathcal{K}} + ch^{1+\alpha\gamma}|p|_{1+\alpha\gamma,\mathcal{K}}.
\end{aligned}$$

Therefore we conclude that

$$|||\boldsymbol{u}-\boldsymbol{u}_h||| + |||p-p_h|||_q \le ch|\boldsymbol{u}|_{2,\mathcal{K}} + ch^{1+\alpha\gamma}|p|_{1+\alpha\gamma,\mathcal{K}}. \tag{20}$$

In order to prove the error estimate for the quantity $\|p-p_h\|_{\mathcal{K}}$ we follow the standard technique, see [9, 13]. Let $\boldsymbol{H}_0^1(\Omega) = \{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega) \mid \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\}$. Since $p-p_h \in L_0^2(\Omega)$ there exists $\boldsymbol{v}_p \in \boldsymbol{H}_0^1(\Omega)$ such that $\nabla\cdot\boldsymbol{v}_p = p - p_h$ and $\|\nabla\boldsymbol{v}_p\|_{\mathcal{K}} \le c\,\|p-p_h\|_{\mathcal{K}}$. Applying integration by parts yields

$$\|p-p_h\|_{\mathcal{K}}^2 = (p-p_h, \nabla\cdot\boldsymbol{v}_p)_{\mathcal{K}} = -(\nabla(p-p_h),\boldsymbol{v}_p)_{\mathcal{K}} + ([p-p_h],\{\boldsymbol{v}_p\})_{\mathcal{F}_i}.$$

Further split this equation

$$\|p-p_h\|_{\mathcal{K}}^2 = I + II \tag{21}$$

with

$$\begin{aligned}
I &= -(\nabla(p-p_h), \boldsymbol{v}_p - \boldsymbol{C_h}\boldsymbol{v}_p)_{\mathcal{K}} + ([p-p_h],\{\boldsymbol{v}_p - \boldsymbol{C_h}\boldsymbol{v}_p\})_{\mathcal{F}_i}, \\
II &= -(\nabla(p-p_h), \boldsymbol{C_h}\boldsymbol{v}_p)_{\mathcal{K}} + ([p-p_h],\{\boldsymbol{C_h}\boldsymbol{v}_p\})_{\mathcal{F}_i}
\end{aligned}$$

and where $\boldsymbol{C}_h$ denotes the vectorial Clément interpolation operator. Firstly observe using the Cauchy-Schwarz inequality, the trace inequality and the approximation properties of $\boldsymbol{C}_h$ that

$$I \le c\,(\|\tilde{h}\nabla(p-p_h)\|_{\mathcal{K}} + \|\tilde{h}^{\frac{1}{2}}[p-p_h]\|_{\mathcal{F}_i})\|\nabla\boldsymbol{v}_p\|_{\mathcal{K}} \le c\,|||p-p_h|||_q\|p-p_h\|_{\mathcal{K}}. \tag{22}$$

Secondly using the consistency of Lemma 4.2 implies

$$\begin{aligned}
II &= -b(p-p_h,\boldsymbol{C_h}\boldsymbol{v}_p) = a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{C_h}\boldsymbol{v}_p) \\
&= (\nabla(\boldsymbol{u}-\boldsymbol{u}_h), \nabla C_h\boldsymbol{v}_p)_{\mathcal{K}} - ([\![\boldsymbol{u}-\boldsymbol{u}_h]\!], \{\nabla\boldsymbol{C}_h\boldsymbol{v}_p\})_{\mathcal{F}} \\
&\le \left(\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{\mathcal{K}}^2 + \|\tilde{h}^{-\frac{1}{2}}[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\|_{\mathcal{F}}^2\right)^{\frac{1}{2}}\left(\|\nabla\boldsymbol{C}_h\boldsymbol{v}_p\|_{\mathcal{K}}^2 + \|\tilde{h}^{\frac{1}{2}}\{\nabla\boldsymbol{C}_h\boldsymbol{v}_p\}\|_{\mathcal{F}}^2\right)^{\frac{1}{2}} \\
&\le c\left(\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{\mathcal{K}} + \|\tilde{h}^{-\frac{1}{2}}[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\|_{\mathcal{F}}\right)\|\nabla\boldsymbol{C}_h\boldsymbol{v}_p\|_{\mathcal{K}} \\
&\le c\left(\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{\mathcal{K}} + \|\tilde{h}^{-\frac{1}{2}}[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\|_{\mathcal{F}}\right)\|p-p_h\|_{\mathcal{K}} \qquad (23)
\end{aligned}$$

using that $\boldsymbol{C}_h\boldsymbol{v}_p \in \boldsymbol{H}_0^1(\Omega)$, the Cauchy-Schwarz inequality, the trace inequality, the $\boldsymbol{H}^1(\Omega)$-stability of $\boldsymbol{C}_h$ and the stability of $\boldsymbol{v}_p$. Combining (20)-(23) leads to the result. □

Remark 5.7 (Optimal L^2-convergence). Optimal convergence in the L^2-norm can be shown using Nitsche's trick. The details are left to the reader.

Proposition 5.8. *Let* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{bs} \times Q_h$ *be the solution of* (13) *with* $Q_h = \alpha\, Q^1_{h,c} \oplus \beta\, Q^0_{h,d}$ *for* $(\alpha, \beta) \in \{(1,0), (0,1), (1,1)\}$*. Then, there holds*

$$h\|\boldsymbol{f} + \Delta \boldsymbol{u}_h - \nabla p_h\|_{\mathcal{K}} + (1-\beta)\|\tilde{h}^{\frac{1}{2}}[\![\nabla \boldsymbol{u}_h]\!]\|_{\mathcal{F}_i} + \|\tilde{h}^{-\frac{1}{2}}\overline{[\![\boldsymbol{u}_h]\!]}\|_{\mathcal{F}} \le ch\|\boldsymbol{f} - \boldsymbol{\pi}_0 \boldsymbol{f}\|_{\mathcal{K}}. \tag{24}$$

Additionally, if $\boldsymbol{f} \in \boldsymbol{H}^1(\mathcal{K})$ *there holds*

$$h\|\boldsymbol{f} + \Delta \boldsymbol{u}_h - \nabla p_h\|_{\mathcal{K}} + (1-\beta)\|\tilde{h}^{\frac{1}{2}}[\![\nabla \boldsymbol{u}_h]\!]\|_{\mathcal{F}_i} + \|\tilde{h}^{-\frac{1}{2}}\overline{[\![\boldsymbol{u}_h]\!]}\|_{\mathcal{F}} \le ch^2\|\nabla \boldsymbol{f}\|_{\mathcal{K}}. \tag{25}$$

Proof. Let $\boldsymbol{w}_h$ be the function defined by Lemma 3.8 with $\boldsymbol{a}_h = h^2(\nabla p_h - \boldsymbol{\pi}_0 \boldsymbol{f} - \Delta \boldsymbol{u}_h)$, $\boldsymbol{b}_h = -\tilde{h}^{-1}\overline{[\![\boldsymbol{u}_h]\!]}\boldsymbol{n}_{\mathtt{F}}$ and $\boldsymbol{c}_h = (1-\beta)\tilde{h}[\![\nabla \boldsymbol{u}_h]\!]$. By its properties, integration by parts, Lemma 3.1, Corollary 3.3 and (2) we get

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{w}_h) &= -(\Delta \boldsymbol{u}_h, \boldsymbol{w}_h)_{\mathcal{K}} + ([\![\nabla \boldsymbol{u}_h]\!], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} - ([\![\boldsymbol{u}_h]\!]\boldsymbol{n}_{\mathtt{F}}, \{\nabla \boldsymbol{w}_h\}\boldsymbol{n}_{\mathtt{F}})_{\mathcal{F}} \\ &= -(\Delta \boldsymbol{u}_h, \boldsymbol{\pi}_0 \boldsymbol{w}_h)_{\mathcal{K}} + ([\![\nabla \boldsymbol{u}_h]\!], \overline{\{\boldsymbol{w}_h\}})_{\mathcal{F}_i} - (\overline{[\![\boldsymbol{u}_h]\!]}\boldsymbol{n}_{\mathtt{F}}, \{\nabla \boldsymbol{w}_h\}\boldsymbol{n}_{\mathtt{F}})_{\mathcal{F}} \\ &= -(\Delta \boldsymbol{u}_h, \boldsymbol{a}_h)_{\mathcal{K}} + (1-\beta)\|\tilde{h}^{\frac{1}{2}}[\![\nabla \boldsymbol{u}_h]\!]\|^2_{\mathcal{F}_i} + \|\tilde{h}^{-\frac{1}{2}}\overline{[\![\boldsymbol{u}_h]\!]}\|^2_{\mathcal{F}}. \end{aligned}$$

On the other hand since $p_h = \alpha\, p_{h,c} + \beta\, p_{h,d}$, using again integration by parts and the properties of the projections we get

$$\begin{aligned} b(p_h, \boldsymbol{w}_h) &= (\nabla p_h, \boldsymbol{w}_h)_{\mathcal{K}} - ([p_h], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} = (\nabla p_h, \boldsymbol{\pi}_0 \boldsymbol{w}_h)_{\mathcal{K}} - \beta([p_{d,h}], \overline{\{\boldsymbol{w}_h\}})_{\mathcal{F}_i} \\ &= (\nabla p_h, \boldsymbol{a}_h)_{\mathcal{K}} - \beta(1-\beta)([p_{d,h}], \tilde{h}[\![\nabla \boldsymbol{u}_h]\!])_{\mathcal{F}_i} = (\nabla p_h, \boldsymbol{a}_h)_{\mathcal{K}} \end{aligned}$$

since $\beta(1-\beta) = 0$ for the considered set of values of β. Moreover for this set we have that $(1-\beta) = (1-\beta)^2$ and since $a(\boldsymbol{u}_h, \boldsymbol{w}_h) + b(p_h, \boldsymbol{w}_h) = (\boldsymbol{f}, \boldsymbol{w}_h)_{\mathcal{K}}$ we get

$$\begin{aligned} &h^2\|\boldsymbol{\pi}_0 \boldsymbol{f} + \Delta \boldsymbol{u}_h - \nabla p_h\|^2_{\mathcal{K}} + (1-\beta)^2\|\tilde{h}^{\frac{1}{2}}[\![\nabla \boldsymbol{u}_h]\!]\|^2_{\mathcal{F}_i} + \|\tilde{h}^{-\frac{1}{2}}\overline{[\![\boldsymbol{u}_h]\!]}\|^2_{\mathcal{F}} \\ &\quad = a(\boldsymbol{u}_h, \boldsymbol{w}_h) + b(p_h, \boldsymbol{w}_h) - (\boldsymbol{\pi}_0 \boldsymbol{f}, \boldsymbol{w}_h)_{\mathcal{K}} = (\boldsymbol{f} - \boldsymbol{\pi}_0 \boldsymbol{f}, \boldsymbol{w}_h)_{\mathcal{K}} \le \|\boldsymbol{f} - \boldsymbol{\pi}_0 \boldsymbol{f}\|_{\mathcal{K}}\|\boldsymbol{w}_h\|_{\mathcal{K}}. \end{aligned}$$

Using further the stability estimate (9) of $\boldsymbol{w}_h$ yields

$$\|\boldsymbol{w}_h\|_{\mathcal{K}} \le ch\left(h^2\|\boldsymbol{\pi}_0 \boldsymbol{f} + \Delta \boldsymbol{u}_h - \nabla p_h\|^2_{\mathcal{K}} + (1-\beta)^2\|\tilde{h}^{-\frac{1}{2}}\overline{[\![\boldsymbol{u}_h]\!]}\|^2_{\mathcal{F}} + \|\tilde{h}^{\frac{1}{2}}[\![\nabla \boldsymbol{u}_h]\!]\|^2_{\mathcal{F}_i}\right)^{\frac{1}{2}}$$

and therefore we get

$$h\|\boldsymbol{\pi}_0 \boldsymbol{f} + \Delta \boldsymbol{u}_h - \nabla p_h\|_{\mathcal{K}} + (1-\beta)\|\tilde{h}^{\frac{1}{2}}[\![\nabla \boldsymbol{u}_h]\!]\|_{\mathcal{F}_i} + \|\tilde{h}^{-\frac{1}{2}}\overline{[\![\boldsymbol{u}_h]\!]}\|_{\mathcal{F}} \le ch\|\boldsymbol{f} - \boldsymbol{\pi}_0 \boldsymbol{f}\|_{\mathcal{K}}.$$

Finally we observe that

$$\|\boldsymbol{f} + \Delta \boldsymbol{u}_h - \nabla p_h\|_{\mathcal{K}} \le \|\boldsymbol{f} - \boldsymbol{\pi}_0 \boldsymbol{f}\|_{\mathcal{K}} + \|\boldsymbol{\pi}_0 \boldsymbol{f} + \Delta \boldsymbol{u}_h - \nabla p_h\|_{\mathcal{K}}$$

which yields (24) and additionally noting that for $\boldsymbol{f} \in \boldsymbol{H}^1(\mathcal{K})$ there holds $\|\boldsymbol{f} - \boldsymbol{\pi}_0 \boldsymbol{f}\|_{\mathcal{K}} \le ch\|\nabla \boldsymbol{f}\|_{\mathcal{K}}$ proves (25). □

As an immediate consequence of the above estimate on the residuals we have

Corollary 5.9. *If* $\boldsymbol{f} \in \boldsymbol{H}^1(\mathcal{K})$ *then*

$$\|\nabla \cdot (\boldsymbol{\sigma}(\boldsymbol{u}, p) - \boldsymbol{\sigma}(\boldsymbol{u}_h, \tilde{p}_h))\|_{\mathcal{K}} \le ch\|\nabla \boldsymbol{f}\|_{\mathcal{K}},$$

where $\boldsymbol{\sigma}(\boldsymbol{u}, p) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) - p\boldsymbol{I}$ *denotes the stress tensor and* $\tilde{p}_h = p_h + \nabla \cdot \boldsymbol{u}_h$ *a modified pressure.*

If $\boldsymbol{f}$ *is piecewise constant, i.e.* $\boldsymbol{f} \in \boldsymbol{V}^0_h$*, then for all valid choices of* α *and* β *there holds,* $\|\overline{[\![\boldsymbol{u}_h]\!]}\|_{\mathcal{F}} = 0$ *and* $\|\nabla \cdot (\boldsymbol{\sigma}(\boldsymbol{u}, p) - \boldsymbol{\sigma}(\boldsymbol{u}_h, \tilde{p}_h))\|_{\mathcal{K}} = 0$*. Moreover when* $\alpha = 1$, $\beta = 0$ *there holds* $\|[\![\nabla \boldsymbol{u}_h]\!]\|_{\mathcal{F}_i} = 0$.

Q_h	h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathcal{K}}$	$\|\|\|\boldsymbol{u}-\boldsymbol{u}_h\|\|\|$	$\|p-p_h\|_{\mathcal{K}}$	$\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\|_{\mathcal{K}}$
$Q^1_{h,c}$	0.1	4.47E-04	1.72E-02	1.95E-03	8.00E-03
	0.05	1.16E-04 (1.95)	8.82E-03 (0.96)	7.21E-04 (1.44)	4.18E-03 (0.94)
	0.025	2.93E-05 (1.98)	4.46E-03 (0.98)	2.54E-04 (1.51)	2.14E-03 (0.97)
	0.0125	7.36E-06 (1.99)	2.24E-03 (0.99)	8.87E-05 (1.52)	1.08E-03 (0.98)
$Q^0_{h,d}$	0.1	2.88E-03	7.24E-02	4.50E-02	2.12E-02
	0.05	7.55E-04 (1.93)	3.76E-02 (0.95)	2.13E-02 (1.08)	1.06E-02 (1.00)
	0.025	1.92E-04 (1.98)	1.90E-02 (0.98)	1.04E-02 (1.04)	5.31E-03 (1.00)
	0.0125	4.81E-05 (1.99)	9.55E-03 (0.99)	5.15E-03 (1.01)	2.65E-03 (1.00)
$Q^1_{h,c}\oplus Q^0_{h,d}$	0.1	3.68E-04	1.50E-02	5.27E-03	6.14E-03
	0.05	9.32E-05 (1.98)	7.59E-03 (0.98)	2.90E-03 (0.86)	3.12E-03 (0.98)
	0.025	2.34E-05 (1.99)	3.82E-03 (0.99)	1.53E-03 (0.93)	1.57E-03 (0.99)
	0.0125	5.85E-06 (2.00)	1.91E-03 (1.00)	7.82E-04 (0.96)	7.86E-04 (1.00)

Table 1. *Smooth problem: Different error quantities of the numerical solution for all three choices of the pressure approximation space with respect to the mesh size h. The quantities in the brackets correspond to the convergence rates.*

Q_h	h	$\|\tilde{h}^{-\frac{1}{2}}[\boldsymbol{u}_h]\|_{\mathcal{F}}$	$\|\tilde{h}^{-\frac{1}{2}}\overline{[\![\boldsymbol{u}_h]\!]}\|_{\mathcal{F}}$	$\|\tilde{h}^{\frac{1}{2}}[\![\nabla\boldsymbol{u}_h]\!]\|_{\mathcal{F}_i}$
$Q^1_{h,c}$	0.1	8.84E-03	1.34E-03	1.94E-03
	0.05	4.55E-03 (0.96)	3.64E-04 (1.88)	2.86E-04 (2.76)
	0.025	2.30E-03 (0.98)	9.38E-05 (1.96)	3.88E-05 (2.88)
	0.0125	1.16E-03 (0.99)	2.37E-05 (1.98)	5.05E-06 (2.94)
$Q^0_{h,d}$	0.1	4.14E-02	1.34E-03	1.54E-01
	0.05	2.20E-02 (0.91)	3.64E-04 (1.88)	8.12E-02 (0.92)
	0.025	1.12E-02 (0.97)	9.38E-05 (1.96)	4.15E-02 (0.97)
	0.0125	5.64E-03 (0.99)	2.37E-05 (1.98)	2.09E-02 (0.99)
$Q^1_{h,c}\oplus Q^0_{h,d}$	0.1	7.87E-03	1.34E-03	1.59E-02
	0.05	4.00E-03 (0.98)	3.64E-04 (1.88)	8.33E-03 (0.93)
	0.025	2.01E-03 (0.99)	9.38E-05 (1.96)	4.26E-03 (0.97)
	0.0125	1.01E-03 (1.00)	2.37E-05 (1.98)	2.16E-03 (0.98)

Table 2. *Smooth problem: Different error quantities of the numerical solution for all three choices of the pressure approximation space with respect to the mesh size h. The quantities in the brackets correspond to the convergence rates.*

Corollary 5.10. *If $\boldsymbol{f}$ is piecewise constant, i.e. $\boldsymbol{f}\in\boldsymbol{V}^0_h$, and $Q_h=Q^1_{h,c}$ then, the discrete solution $\boldsymbol{u}_h$ and p_h of* (13) *satisfies the following local conservation property*

$$-\int_{\partial\kappa}\nabla\boldsymbol{u}_h\boldsymbol{n}_\kappa\,ds+\int_{\partial\kappa}p_h\boldsymbol{n}_\kappa\,ds=\int_\kappa\boldsymbol{f}\,dx$$

for all $\kappa\in\mathcal{K}$ and where $\boldsymbol{n}_\kappa$ denotes the outer normal of κ.

6. Numerical tests

Let us introduce the numerical examples presented in this section. We consider two numerical tests proposed in [13].

i) *Problem with smooth solution*
Consider problem (12) with $\Omega=(0,1)^2$ and $\boldsymbol{f}(\boldsymbol{x})$ imposed such that the

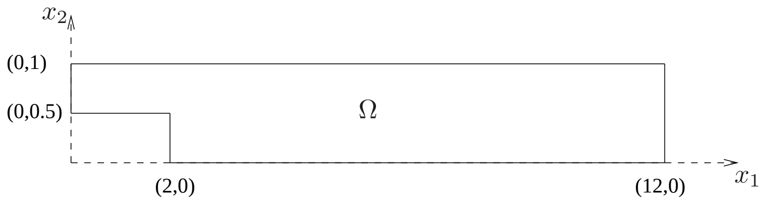

FIGURE 1. *Geometry of the backstep channel problem.*

exact solution is given by

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} (x_1^4 - 2x_1^3 + x_1^2)(4x_2^3 - 6x_2^2 + 2x_2) \\ -(4x_1^3 - 6x_1^2 + 2x_1)(x_2^4 - 2x_2^3 + x_2^2) \end{pmatrix}, \quad p(\boldsymbol{x}) = x_1 + x_2 - 1.$$

Observe that $(\boldsymbol{u}, p)$ satisfies the regularity assumption and thus Theorem 5.6 and Proposition 5.8 are valid. A sequence of structured meshes is considered.

ii) *Backstep channel problem*
Consider problem (12) with $\boldsymbol{f} = \boldsymbol{0}$ and

$$\boldsymbol{g}(\boldsymbol{x}) = \begin{cases} (8(1-x_2)(x_2-0.5), 0)^T & \text{if } x_1 = 0, \\ ((1-x_2)x_2, 0)^T & \text{if } x_1 = 12, \\ \boldsymbol{0} & \text{otherwise,} \end{cases}$$

on the domain defined by Figure 1. The domain is not convex and thus the solution does not lie in $\boldsymbol{H}^2(\Omega)$. In consequence Theorem 5.6 is no longer valid. But observe that Proposition 5.8 is independent of the geometry resp. regularity assumption and therefore Proposition 5.8 still holds. A sequence of unstructured and globally uniform meshes is considered.

We consider the approximations defined by (13) using a pressure approximation space $Q_h = \alpha\, Q_{h,c}^1 \oplus \beta\, Q_{h,d}^0$ for $(\alpha, \beta) \in \{(1,0), (0,1), (1,1)\}$. For the computations we use FreeFem++ [12].

6.1. **Smooth problem.** The convergence results for test problem i) with all three choices of the pressure approximation space are illustrated in Table 1 and 2. We observe the optimal convergence as predicted by Theorem 5.6 and the super convergence of Proposition 5.8.

6.2. **Backstep channel problem.** The convergence results for test problem ii) with all three choices of the pressure approximation space is illustrated in Table 3. Since the exact solution is not known only the "non-conforming" error-quantities are given. Observe that Proposition 5.8 is still valid, in contrast to Theorem 5.6 since Ω is non-convex, and since $\boldsymbol{f} = 0$ the quantity $\|\tilde{h}^{-\frac{1}{2}} \overline{[\![\boldsymbol{u}_h]\!]}\|_{\mathcal{F}}$ is for any choice of pressure space equal to zero (machine precision).

ACKNOWLEDGEMENTS

The authors acknowledge the financial support provided through the Swiss National Science Foundation under grant 200021 − 113304.

Q_h	h	$\\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\\|_{\mathcal{K}}$	$\\|\tilde{h}^{-\frac{1}{2}}[\boldsymbol{u}_h]\\|_{\mathcal{F}}$	$\\|\tilde{h}^{-\frac{1}{2}}[\![\boldsymbol{u}_h]\!]\\|_{\mathcal{F}}$	$\\|\tilde{h}^{\frac{1}{2}}[\![\nabla\boldsymbol{u}_h]\!]\\|_{\mathcal{F}_i}$
$Q^1_{h,c}$	0.25	6.63E-01	7.95E-01	1.03E+00	$0^\star$
	0.125	3.99E-01 (0.73)	4.39E-01 (0.86)	6.00E-01 (0.78)	$0^\star$
	0.0625	2.05E-01 (0.96)	2.54E-01 (0.79)	3.27E-01 (0.88)	$0^\star$
	0.03125	1.19E-01 (0.78)	1.27E-01 (1.01)	1.73E-01 (0.91)	$0^\star$
$Q^0_{h,d}$	0.25	$0^\star$	9.78E-01	1.21E+00	2.61E+00
	0.125	$0^\star$	6.70E-01 (0.54)	8.32E-01 (0.54)	1.86E+00 (0.49)
	0.0625	$0^\star$	3.98E-01 (0.75)	4.84E-01 (0.78)	1.03E+00 (0.85)
	0.03125	$0^\star$	2.06E-01 (0.93)	2.54E-01 (0.93)	5.61E-01 (0.87)
$Q^1_{h,c}\oplus Q^0_{h,d}$	0.25	6.67E-01	5.91E-01	7.36E-01	1.82E+00
	0.125	3.57E-01 (0.90)	2.88E-01 (1.04)	3.83E-01 (0.94)	1.09E+00 (0.73)
	0.0625	1.87E-01 (0.93)	1.79E-01 (0.68)	2.19E-01 (0.81)	5.57E-01 (0.97)
	0.03125	9.28E-02 (1.01)	8.82E-02 (1.02)	1.17E-01 (0.90)	3.07E-01 (0.86)

Table 3. *Backstep channel problem: Different error quantities of the numerical solution for all three choices of the pressure approximation space with respect to the mesh size h. The quantities in the brackets correspond to the convergence rates and $0^\star$ corresponds to zero in machine precision.*

References

[1] S. C. Brenner. Poincaré-Friedrichs inequalities for piecewise H^1 functions. *SIAM J. Numer. Anal.*, 41(1):306–324 (electronic), 2003.

[2] S. C. Brenner and L. Owens. A weakly over-penalized non-symmetric interior penalty method. *JNAIAM J. Numer. Anal. Ind. Appl. Math.*, 2(1-2):35–48, 2007.

[3] F. Brezzi and L. D. Marini. Bubble stabilization of discontinuous Galerkin methods. In W. Fitzgibbon, R. Hoppe, J. Periaux, O. Pironneau, and Y. Vassilevski, editors, *Advances in numerical mathematics, Proc. International Conference on the occasion of the 60th birthday of Y.A. Kuznetsov*, pages 25–36. Institute of Numerical Mathematics of The Russian Academy of Sciences, Moscow, 2006.

[4] E. Burman and B. Stamm. Low order discontinuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.*, 47(1):508–533, 2008.

[5] E. Burman and B. Stamm. Symmetric and non-symmetric discontinuous Galerkin methods stabilized using bubble enrichment. *C. R. Math. Acad. Sci. Paris*, 346(1-2):103–106, 2008.

[6] P. G. Ciarlet. *The finite element method for elliptic problems*, volume 40 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)].

[7] P. Clément. Approximation by finite element functions using local regularization. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. RAIRO Analyse Numérique*, 9(R-2):77–84, 1975.

[8] B. Cockburn, G. Kanschat, D. Schötzau, and C. Schwab. Local discontinuous Galerkin methods for the Stokes system. *SIAM J. Numer. Anal.*, 40(1):319–343 (electronic), 2002.

[9] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.

[10] V. Girault, B. Rivière, and M. F. Wheeler. A discontinuous Galerkin method with nonoverlapping domain decomposition for the Stokes and Navier-Stokes problems. *Math. Comp.*, 74(249):53–84 (electronic), 2005.

[11] P. Hansbo and M. G. Larson. Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method. *Comput. Methods Appl. Mech. Engrg.*, 191(17-18):1895–1908, 2002.

[12] F. Hecht, O. Pironneau, A. L. Hyaric, and K. Ohtsuka. *FreeFEM++ Manual*, 2008.

[13] B. Rivière and V. Girault. Discontinuous finite element methods for incompressible flows on subdomains with non-matching interfaces. *Comput. Methods Appl. Mech. Engrg.*, 195(25-28):3274–3292, 2006.

[14] A. Toselli. *hp* discontinuous Galerkin approximations for the Stokes problem. *Math. Models Methods Appl. Sci.*, 12(11):1565–1597, 2002.

Erik Burman, Department of Mathematics, Mantell Building, University of Sussex, Falmer, Brighton, BN1 9RF, United Kingdom
E-mail address: `E.N.Burman@sussex.ac.uk`

Benjamin Stamm, Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, United States
E-mail address: `Benjamin_Stamm@Brown.edu`